Quantum mechanics of measurement

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An analysis of quantum measurement is presented that relies on an information-theoretic description of quantum entanglement. In a consistent quantum information theory of entanglement, entropies (uncertainties) *conditional* on measurement outcomes can be *negative*, implying that measurement can be described via unitary, entropy-conserving, interactions, while still producing randomness in a measurement device. In such a framework, quantum measurement is not accompanied by a wave-function collapse, or a quantum jump. The theory is applied to the measurement of incompatible variables, giving rise to a stronger entropic uncertainty relation than heretofore known. It is also applied to standard quantum measurement situations such as the Stern-Gerlach and double-slit experiments to illustrate how randomness, inherent in the conventional quantum probabilities, arises in a unitary framework. Finally, the present view clarifies the relationship between classical and quantum concepts.

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I. INTRODUCTION

For seventy years it has remained a mystery how quantum measurement can be probabilistic in nature, and thus be accompanied by the creation of randomness or uncertainty, while at the same time being described by unitary evolution. This apparent contradiction has cast serious doubts on the very foundations of quantum mechanics. Meanwhile, the equations and predictions of the purportedly flawed theory enjoy unbridled, unequivocal success. In this paper, we present an information-theoretic description of quantum measurement which sheds new light on this long-standing question. This description, in terms of the quantum information theory (QIT) introduced by us recently [1, 2], implies that the (conditional) quantum entropy of an entangled subsystem can be *negative*, in contrast to its classical counterpart. As we outline below, this allows for the creation of entropy in the measurement device which is counterbalanced by the negative entropy of the quantum system itself, resulting in the conservation of entropy in the measurement process. Consequently, the probabilistic nature of quantum mechanics can be shown to follow from a completely consistent, unitary, description of measurement.

Our model does not require the quantum system to be coupled to a macroscopic—uncontrollable—environment, and is therefore distinct from the environment-induced decoherence model, one of the prevalent contemporary views of quantum measurement [3]. As shown below, the view advocated here only insists on the "self-consistency" of the measurement device while abandoning the 100% correlation between the device and the measured quantum system which is a cornerstone of decoherence models. Also, while the information-theoretic interpretation suggests that the universe exists in a *superposition* of quantum states, all of quantum phenomenology is explained armed only with one, rather than many, such universes.

A more detailed investigation of the measurement process reveals that the collapse of the wavefunction is an illusion, brought about by the observation of part of a composite system that is quantum entangled and thus *inseparable*. Rather than collapsing, the wavefunction of a measured system becomes entangled with the wavefunction of the measurement device. If prepared in a superposition of eigenstates, the measured system is *not* reduced to one of its eigenstates. In other words, a quantum jump does *not* occur. That this must be the case has of course been suspected for a long time, and it certainly is implicit in the quantum eraser experiments on which we shall comment below. Here we show that this feature emerges naturally if quantum entanglement is properly described in the language of QIT. Furthermore, due to the absence of a collapse of the wavefunction, our unitary description implies that quantum measurement is inherently *reversible*, overturning the common view. However, in an experiment where quantum entanglement is transferred to a macroscopic "pointer" variable (as is essential for classical observers) the reversibility is obscured by the *practical* impossibility of keeping track of all the atoms involved in the unitary transformation, rendering the measurement as irreversible as the thermodynamics of gases[1]. Thus, as suggested earlier by Peres [4], the apparent irreversibility of quantum measurement can be understood entirely in classical terms.

In the next section, we briefly review the current state of quantum measurement theory, with emphasis on the standard von Neumann theory of measurement. In Section III, we outline those features of the quantum information theory introduced in [1, 2] which apply to quantum measurement, and point out the singular importance of negative entropy in quantum entanglement. We also focus on the relation between entanglement and inseparability in this theory. In Section IV we then proceed with a microscopic description of the unitary physical measurement process as anticipated by von Neumann, but properly interpreted within QIT. We focus on the measurement of incompatible variables in Section V and show how one of the milestones of quantum physics, the uncertainty relation, emerges naturally from our construction. Alternatively, this Section can be read as describing unitary quantum measurement more formally, implying some of the well-known relations of conventional quantum mechanics. Section VI discusses new insights into the interpretation of quantum mechanics brought about by this information-theoretic analysis. There, we investigate the relationship between classical and quantum variables and propose a simple resolution to the "Schrödinger-cat" paradox. Also, we comment on the origin of the complementarity principle and the duality between waves and particles. We offer our conclusions in Section VII. Finally, Appendix A illustrates the interpretation of standard experiments of quantum mechanics within our framework. There, we consider the basic Stern-Gerlach setup and "quantum erasure" in the standard double-slit experiment.

II. THEORY OF MEASUREMENT

The theory of measurement occupies a central role in quantum physics and has undergone a number of conceptual revolutions. Those range from the probabilistic interpretation of quantum mechanics by Born and the Copenhagen interpretation championed by Bohr (see e.g. [5]), over von Neumann's seminal contribution in the "*Grundlagen*" [6] to more modern interpretations such as

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[1] This last conclusion is reached in environment-induced decoherence models as well, since there is no qualitative difference between an environment and a large number of degrees of freedom belonging to a macroscopic measurement device.

Everett's [7, 8], Cramer's [9], and Zurek's [3].

Central to all these treatments is the problem of the collapse of the wavefunction, or state vector. To illustrate this process, consider for example the measurement of an electron, described by the wavefunction $\Psi(q)$ where q is the coordinate of the electron. Further, let the measurement device be characterized initially by its eigenfunction $\phi_0(\xi)$, where ξ may summarize the coordinates of the device. Before measurement, i.e., before the electron interacts with the measurement device, the system is described by the wavefunction

$$\Psi(q)\phi_0(\xi)\ . \tag{2.1}$$

After the interaction, the wavefunction is a superposition of the eigenfunctions of electron and measurement device

$$\sum_n \psi_n(q)\phi_n(\xi)\ . \tag{2.2}$$

Following orthodox measurement theory, the classical nature of the measurement apparatus implies that after measurement the "pointer" variable ξ takes on a well-defined value at each point in time; the wavefunction, as it turns out, is thus *not* given by the entire sum in (2.2) but rather by the single term

$$\psi_n(q)\phi_n(\xi)\ . \tag{2.3}$$

The wavefunction (2.2) is said to have collapsed to (2.3).

A cornerstone of the Copenhagen interpretation of measurement was precisely this collapse, due to the interaction of a quantum object with a macroscopic, *classical*, measurement device. The crucial step to describe the measurement process as an interaction of two *quantum* systems [as is implicit in (2.2)] was made by von Neumann [6], who recognized that an interaction between a classical and a quantum system cannot be part of a consistent quantum theory. In his *Grundlagen*, he therefore proceeded to decompose the quantum measurement into *two* fundamental stages. The first stage (termed "von Neumann measurement") gives rise to the wavefunction (2.2). The second stage (which von Neumann termed "observation" of the measurement) involves the collapse described above, i.e., the transition from (2.2) to (2.3).

We now proceed to describe the first stage in more detail. For ease of notation, let us recast this problem into the language of state vectors instead. The first stage involves the interaction of the quantum system Q with the measurement device (or "ancilla") A. Both the quantum system and the ancilla are fully determined by their state vector, yet, let us assume that the state of Q (described by state vector $|x\rangle$) is unknown whereas the state of the ancilla is prepared in a special state $|0\rangle$, say. The state vector of the combined system $|QA\rangle$ before measurement then is

$$|\Psi_{t=0}\rangle = |x\rangle|0\rangle \equiv |x,0\rangle\ . \tag{2.4}$$

The von Neumann measurement is described by the unitary evolution of QA via the interaction Hamiltonian

$$\hat{H} = -\hat{X}_Q\hat{P}_A\ , \tag{2.5}$$

operating on the product space of Q and A. Here, $\hat{X}_Q$ is the observable to be measured, and $\hat{P}_A$ the operator *conjugate* to the degree of freedom of A that will reflect the result of the measurement. We now obtain for the state vector $|QA\rangle$ after measurement (e.g. at $t=1$, putting $\hbar=1$)

$$|\Psi_{t=1}\rangle = e^{i\hat{X}_Q\hat{P}_A}|x,0\rangle = e^{ix\hat{P}_A}|x,0\rangle = |x,x\rangle\ . \tag{2.6}$$

Thus, the pointer in A that previously pointed to zero now also points to the position x that Q is in. According to von Neumann, this simple operation reflects the *correlation* between Q and A introduced by the measurement. In general, this unitary operation rather introduces *entanglement*, which is beyond the classical concept of correlations. In fact, the creation of entanglement in a von Neumann measurement[2] is *generic*. This is illustrated for typical measurement situations in Appendix A.

The second stage in von Neumann's theory of measurement, the *observation* of the pointer variable by a conscious observer (or a mechanical device with memory), is the key problem of measurement theory and the central object of this paper. Historically, this conundrum is usually couched into the question: "At what point does the possibility of an outcome change into actuality?" In the interpretation of this stage, von Neumann finally conceded to Bohr, who maintained that the "observing" operation (stage two), now distinct from the "measuring" process (stage one), is *irreversible* and *non-causal.* At first glance, there appears to be no escape from this conclusion, as a pure state (a superposition) seems to evolve into a mixed state (describing all possible outcomes), a process that cannot be described by a unitary operation. This becomes more evident if we apply the unitary operation described above to an initial quantum state which is in a quantum superposition:

$$|\Psi_{t=0}\rangle = |x+y,0\rangle\ . \tag{2.7}$$

Then, the linearity of quantum mechanics implies that

$$|\Psi_{t=1}\rangle = e^{i\hat{X}_Q\hat{P}_M}\Big(|x,0\rangle + |y,0\rangle\Big) = |x,x\rangle + |y,y\rangle \tag{2.8}$$

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[2] A general measurement can be described using a positive-operator-valued measure (POVM), based on the decomposition of the identity operator into positive operators on the Hilbert space [10]. The von Neumann measurement is a special case in which the positive operators are the orthogonal projection operators $|X_Q\rangle\langle X_Q|$ (which sum to identity because of the closure relation). The restriction to a simple von Neumann measurement, however, is sufficient for our purposes since a POVM can always be described as a von Neumann measurement in an extended Hilbert space.

which is still a pure state. However, it does not reflect classical correlations between Q and A (as would the state $|x+y, x+y\rangle$) but rather *quantum entanglement*. This realization is the content of the celebrated quantum non-cloning theorem [11]. Just like the wavefunction (2.2), the state vector (2.8) cannot describe the result of the observation of the pointer, as the pointer is classical and takes on definite values. Thus, a measurement will reveal A to be in the state $|x\rangle$ *or* $|y\rangle$, the sum (2.8) will appear to have collapsed, and a "completely known" (fully described) quantum object seems to have evolved into one of several possible outcomes. This recurrent problem forced von Neumann to introduce a process *different* from unitary evolution to describe the second stage in quantum measurement, the *observation* of A in the entangled system QA. While he showed that the boundary between the observed system QA and the observer can be placed arbitrarily, he still concluded that "observation" must ultimately take place. Reluctantly, he suggested that the collapse of the wavepacket had to occur in the observer's brain, thereby allowing the concept of consciousness to enter in his description of measurement [6, 12, 13].

To this date, there is no unanimous agreement on a solution to this problem. A promising attempt at unraveling the mystery was presented by Everett [7]. In his interpretation, measurement is described exactly as outlined above, only the second stage never takes place. Rather, the different terms in the sum (2.2) or (2.8) are interpreted as the "records" of (conscious or mechanical) observers, each recording possible versions of reality, while only one particular term is available for one observer in a particular instantiation. The sum has been interpreted by DeWitt [8] as the wavefunction of a universe constantly branching at each quantum event. While internally consistent, the Everett–DeWitt interpretation suffers from the burden of unprovable *ad hoc* assumptions. Interesting from the point of view advanced here are the formulations of Peres [4] and Zurek [3], generally referred to as environment-induced decoherence models. In their approach, mixed states are obtained from pure states by tracing over either the measurement apparatus (for example because it has many uncontrollable degrees of freedom) or a macroscopic environment (which absorbs the quantum phases because it involves enormously numerous random degrees of freedom). The underlying idea thus is that the loss of information in a macroscopic system is responsible for the creation of entropy in a measurement. While accounting for the apparently irreversible character of quantum measurement, this approach does not address the issue of the collapse, nor does it provide a satisfying explanation for the Schrödinger-cat paradox (see, e.g., [14]). Another interesting attempt is due to Cramer [9], who invokes the exchange of retarded and advanced waves between elements of a measurement situation in the second stage of measurement. The difficulty to describe quantum measurement as unitary evolution is affecting areas of physics as diverse as black holes and quantum optics. Attempts at tackling the problem range from giving up unitarity in quantum mechanics to understand the production of entropy in Hawking radiation [15], to describing quantum decoherence via a non-Liouvillian equation [16]. Most recently, it was suggested that using DNA as a microscopic measuring device [14] (to record the absorption of ultraviolet photons) would reveal that "[...] even the most prominent nonorthodox models of quantum mechanics have nontrivial difficulties" if no essential role is ascribed to a conscious observer!

Historically, it appears that the failure to understand von Neumann's second stage is rooted in a misunderstanding of the correlations introduced by the first stage. In fact, it was only three years *after* the appearance of the *Grundlagen* that Einstein, Podolsky, and Rosen (EPR) [17] pointed out the peculiarities of a wavefunction such as (2.8), now known as the wavefunction of an EPR entangled state. As we shall see in the next Section, correlations inherent in such a state cannot be understood via classical concepts, as the state so created is not separable. The observation of only a *part* of such a system effects the appearance of probabilities (in a subsystem) when in fact none such are present (in the combined system). The second stage of measurement can be understood *without* recourse to non-unitary time-evolution or the intervention of consciousness, within the language of the quantum information theory introduced recently [1, 2].

III. QUANTUM INFORMATION THEORY

In the standard information theoretical treatment of quantum measurement, classical (Shannon) information theory [18] is applied to probabilities derived from quantum mechanics. More precisely, the quantum probabilities of the different outcomes of the measurement of a quantum state are used to calculate the tradeoff between entropy and information that accompanies the measurement [19]. However, this treatment is incomplete, as the quantum probabilities entering Shannon theory are devoid of the phase information which characterizes quantum mechanical superpositions. To be consistent, quantum information theory needs to be based on density matrices only, rather than on probability distributions.

Let us summarize the unified information-theoretical description of correlation and entanglement that was introduced in Ref. [1, 2]. This theory parallels classical (Shannon) information theory, but extends it to the quantum regime. A quantum system A, described by a density matrix ρ_A, has von Neumann entropy

$$S(A) = -\mathrm{Tr}_A[\rho_A \log \rho_A] \tag{3.1}$$

where Tr_A denotes the trace over the degrees of freedom associated with A. If ρ_A is expressed in a diagonal basis, i.e., $\rho_A = \sum_a p(a)|a\rangle\langle a|$, the von Neumann entropy is equal to the classical (Shannon-Boltzmann-Gibbs) entropy

$$H(A) = -\sum_a p(a) \log p(a) \ . \tag{3.2}$$

An important property of the von Neumann entropy $S(A)$ is that it remains constant when the system A undergoes a unitary transformation. This is analogous to the Boltzmann entropy remaining constant under a reversible transformation in classical thermodynamics. As quantum mechanics only allows unitary time-evolution, the von Neumann entropy of any isolated system remains constant in time.

The substitution of probabilities (in classical information theory) by density matrices (in quantum information theory) becomes crucial when considering composite systems, such as a bipartite system AB. Indeed, the density matrix ρ_{AB} of the entire system can in general not be written as a diagonal matrix, if changes of basis are performed on the variables associated to A and B *separately*. (Of course, ρ_{AB} can always be diagonalized by applying a change of variables to a joint basis.) The composite system AB is associated with a von Neumann entropy

$$S(AB) = -\mathrm{Tr}_{AB}[\rho_{AB} \log \rho_{AB}] \tag{3.3}$$

Now, in order to analyze a measurement situation, we need to consider a *conditional* quantum entropy $S(A|B)$, which describes the entropy of A *knowing* B. Let $S(A|B)$ therefore denote the von Neumann entropy of A *conditional* on B, and be given by

$$S(A|B) = -\mathrm{Tr}_{AB}[\rho_{AB} \log \rho_{A|B}] \tag{3.4}$$

with

$$\rho_{A|B} = \lim_{n\to\infty} \left[\rho_{AB}^{1/n} (\mathbf{1}_A \otimes \rho_B)^{-1/n}\right]^n \tag{3.5}$$

the "conditional" density matrix defined in [1]. Here, $\otimes$ stands for the tensor product in the joint Hilbert space and $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ denotes a "marginal" (or reduced) density matrix, obtained by a partial trace over the variables associated with A only. The conditional density matrix defined here is just the quantum analogue of the conditional probability $p(a|b) = p(a,b)/p(b)$ in classical information theory and reduces to it in a classical situation (i.e., when the density matrix is diagonal). In the case that ρ_{AB} and $(\mathbf{1}_A \otimes \rho_B)$ commute, Eq. (3.5) simply reduces to

$$\rho_{A|B} = \rho_{AB}(\mathbf{1}_A \otimes \rho_B)^{-1} \ . \tag{3.6}$$

Using Eqs. (3.4) and (3.5), it can be checked that the total entropy decomposes as

$$S(AB) = S(A) + S(B|A) = S(B) + S(A|B) \ , \tag{3.7}$$

in perfect analogy with the equations relating classical entropies. We also define a quantum *mutual* entropy

$$S(A{:}B) = -\mathrm{Tr}_{AB}[\rho_{AB} \log \rho_{A:B}] \tag{3.8}$$

with

$$\rho_{A:B} = \lim_{n\to\infty} \left[(\rho_A \otimes \rho_B)^{1/n} \rho_{AB}^{-1/n}\right]^n \ , \tag{3.9}$$

which reduces to

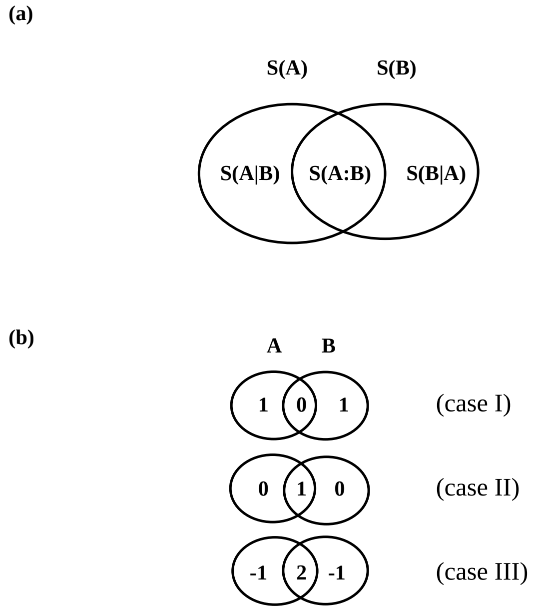
$$\rho_{A:B} = (\rho_A \otimes \rho_B)\rho_{AB}^{-1} \tag{3.10}$$

for commuting matrices. Using Eqs. (3.8) and (3.9), the quantum mutual entropy can be written as

$$S(A{:}B) = S(A) + S(B) - S(AB) \tag{3.11}$$

and is interpreted as the "shared" entropy between A and B. Eqs. (3.7) and (3.11) precisely parallel the classical relations, and validate the definitions (3.5) and (3.9). The relations between $S(A)$, $S(B)$, $S(AB)$, $S(A|B)$, $S(B|A)$, and $S(A{:}B)$ are conveniently summarized by a Venn-like entropy *diagram*, as shown in Fig. 1a.

FIG. 1. (a) General entropy diagram for a quantum composite system AB. (b) Entropy diagrams for three cases of two spin-1/2 particles: (I) independent, (II) classically correlated, and (III) quantum EPR-entangled.



As mentioned earlier, in spite of the apparent similarity between the quantum definitions for $S(A|B)$ or $S(A{:}B)$ and their classical counterparts, dealing with matrices (rather than scalars) opens up a quantum realm for information theory that is inaccessible to classical physics. The crucial point is that, while a conditional probability is a probability distribution (*i.e.*, $0 \leq p(a|b) \leq 1$), its quantum analogue $\rho_{A|B}$ is *not* a density matrix. In general, $\rho_{A|B}$ is a positive Hermitian matrix in the joint Hilbert space, but it can have eigenvalues exceeding one, and consequently, the associated conditional entropy

$S(A|B)$ can be negative. In classical information theory, a conditional entropy $H(A|B)$ is always non-negative. This is in agreement with common sense, since the classical entropy of a composite system AB cannot be lower than the entropy of any subsystem A or B. More precisely, for classical entropies, we have the basic inequality,

$$\max[H(A), H(B)] \leq H(AB) \leq H(A) + H(B) \quad (3.12)$$

where the upper bound is reached for independent subsystems, while the lower bound corresponds to maximally correlated subsystems and implies $H(A|B) \geq 0$, $H(B|A) \geq 0$. In contrast, the equivalent inequality (due to Araki and Lieb [20]) for quantum entropies becomes

$$|S(A) - S(B)| \leq S(AB) \leq S(A) + S(B) \quad (3.13)$$

where the lower bound can be *lower* than the classical one, implying that $S(A|B)$ or $S(B|A)$ can be negative. This well-known non-monotonicity of quantum entropies follows naturally in our matrix-based formalism from the fact that $\rho_{A|B}$ can have eigenvalues larger than one. The situation where $S(A) > S(AB)$ or $S(B) > S(AB)$ occurs in the case of quantum entanglement.

As an illustration, it is instructive to consider three simple cases of two spin-1/2 particles with entropy[3] $S(A) = S(B) = 1$. In our first case I, let the particles be independent, each one being described by the density matrix

$$\rho_A = \rho_B = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \quad (3.14)$$

Then, the entire system has $\rho_{AB} = \rho_A \otimes \rho_B$, so that the total entropy is $S(AB) = 2$, while each system carries one bit of entropy (see Fig. 1b). Also, we have $\rho_{A|B} = \rho_A \otimes \mathbf{1}_B$ and $\rho_{B|A} = \mathbf{1}_A \otimes \rho_B$, implying that $S(A|B) = S(A)$ and $S(B|A) = S(B)$. In our next case II, let the two particles be fully (classically) correlated, so that

$$\rho_{AB} = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) \; . \quad (3.15)$$

This is a uniform mixture, with the two particles always in the same state (i.e., classically correlated). The respective entropies are shown in Fig. 1b. Our last case III is quantum entanglement, and corresponds physically to the situation which appears when a singlet state is created by the decay of a spin-0 particle into two spin-1/2 particles (creating an "EPR-pair"). Such a system is described by the EPR wave function[4]

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \; . \quad (3.16)$$

Here, $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$, so that we have $S(AB) = 0$, as expected for a pure quantum state. By taking a partial trace of ρ_{AB}, we see that both subsystems A and B are in a mixed state

$$\rho_A = \rho_B = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) \; , \quad (3.17)$$

as in cases I and II. Such mixed states have positive entropy, yet, the combined entropy is zero in this case. Then, the conditional entropies are forced to be *negative*, $S(A|B) = S(B|A) = -1$, whereas the mutual entropy $S(A:B) = 2$ (this is illustrated in Fig. 1b). This can be verified by straightforward evaluation. In general, conditional entropies are negative for *any* isolated ($S = 0$) entangled quantum system. Note further that the EPR entanglement constraint $[S(AB) = 0]$ for an EPR pair arises from the fact that it is created via a unitary transformation from a system initially in a zero entropy pure state (the decay of the spin-0 particle). This constraint implies that only one of the three entropies $S(A|B)$, $S(B|A)$, and $S(A:B)$, is an independent variable. In other words, the entropy diagram of *any* pure entangled bipartite system can only be a multiple of that of case III in Fig. 1b. This situation violates the classical inequalities [Eq. (3.12)] that relate Shannon entropies, and therefore corresponds to a purely quantum situation, while cases I and II are classically allowed [1, 2]. In this sense, the matrix-based framework presented above must be seen as an extension of Shannon theory: it describes all the situations allowed classically (from case I to case II), but extends to entanglement (case III).

The appearance of "unclassical" (> 1) eigenvalues in the conditional density matrix of entangled states can be related to quantum non-separability and the violation of entropic Bell inequalities, as shown elsewhere [21]. As far as the separability of a *pure* state is concerned, it is straighforward to check that the non-negativity of the conditional entropy is a necessary and sufficient condition for separability. The separability of *mixed* states, on the other hand, presents a more difficult problem. First, the concavity of $S(A|B)$ in ρ_{AB}, a property related to strong subadditivity of quantum entropies, implies that any separable state [22]

$$\rho_{AB} = \sum_k w_k \, \rho_A^{(k)} \otimes \rho_B^{(k)} \qquad (\text{with } \sum_k w_k = 1) \quad (3.18)$$

is associated with a non-negative conditional entropy $S(A|B)$. (The converse is not true.) Indeed, each product component $\rho_A^{(k)} \otimes \rho_B^{(k)}$ of a separable state is associated with the conditional density matrix

$$\rho_{A|B}^{(k)} = \rho_A^{(k)} \otimes \mathbf{1}_B \quad (3.19)$$

so that we have

$$S(A|B) \geq \sum_k w_k S(\rho_A^{(k)}) \geq 0 \; . \quad (3.20)$$

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[3] Throughout this paper we take logarithms to the base two, such that entropies are expressed in *bits*.

[4] The state in (3.16) is in fact one of the *Bell* states, which are a generalization of the EPR state.

This shows that the non-negativity of conditional entropies is a *necessary* condition for separability. This condition is shown to be equivalent to the non-violation of entropic Bell inequalities in Ref. [21]. Secondly, it is easy to check from Eq. (3.4) that, if $S(A|B)$ is negative, $\rho_{A|B}$ must admit at least one "non-classical" eigenvalue, *i.e.*, an eigenvalue exceeding one. This results from the fact that $\mathrm{Tr}(\rho\sigma) \geq 0$ if ρ and σ are positive (Hermitian) matrices. We have checked that *all* the eigenvalues of $\rho_{A|B}$ and $\rho_{B|A}$ are ≤ 1 for randomly generated separable density matrices [of the form Eq. (3.18)], which suggests the conjecture that the "classicality" of the spectrum of $\rho_{A|B}$ is a (strong) necessary condition for separability[5].

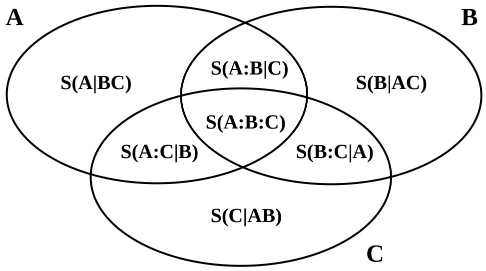
For example, this criterion can be applied to two spin-1/2 particles in a Werner state, that is a mixture of a singlet fraction x and a random fraction $(1-x)$, as recently examined by Peres [23]. The density matrix of this state is given by

$$\rho_{AB} = \begin{pmatrix} (1-x)/4 & 0 & 0 & 0 \\ 0 & (1+x)/4 & -x/2 & 0 \\ 0 & -x/2 & (1+x)/4 & 0 \\ 0 & 0 & 0 & (1-x)/4 \end{pmatrix} \tag{3.21}$$

A simple calculation shows that $\rho_{A|B}$ admits three eigenvalues equal to $(1-x)/2$, and a fourth equal to $(1+3x)/2$. The above separability criterion is thus fulfilled when this fourth eigenvalue does not exceed 1, that is for $x \leq 1/3$. Therefore, for this particular case, our condition simply reduces to Peres' condition based on the positivity of the partial transpose of ρ_{AB}.[6] (It happens to be a sufficient condition for a 2×2 Hilbert space.) We have checked, however, that our criterion is distinct from Peres' in general, opening the possibility that it could be a stronger necessary (or perhaps sufficient) condition for separability in a Hilbert space of arbitrary dimensions. Further work will be devoted to this question.

The description of quantum entanglement within this information-theoretic framework turns out to be very powerful when considering tripartite – or more generally multipartite– quantum systems. Indeed, it is possible to extend to the quantum regime the various classical entropies that are defined in the Shannon information-theoretic treatment of a multipartite system. This accounts for example for the emergence of classical correlation from quantum entanglement in a tripartite (or larger) system. Also, the quantum analogues of all the fundamental relations between classical entropies (such as the chain rules for entropies and mutual entropies) hold in quantum information theory and have the same intuitive interpretation. Let us first consider a simple diagrammatic way of representing quantum entropies involved in a tripartite system ABC, as shown in Figure 2.

FIG. 2. Ternary entropy Venn-diagram for a general tripartite system ABC. The component entropies are defined in the text.



The conditional entropies $S(A|BC)$, $S(B|AC)$, and $S(C|AB)$ are a straightforward generalization of conditional entropies in a bipartite system, that is $S(A|BC) = S(ABC) - S(BC)$, etc. The entropies $S(A{:}B|C)$, $S(A{:}C|B)$, and $S(B{:}C|A)$ correspond to conditional mutual entropies, *i.e.* the mutual entropy between two of the subsystems when the third is known. In perfect analogy with the classical definition, one can write,

$$\begin{aligned} S(A{:}B|C) &= S(A|C) - S(A|BC) \\ &= S(AC) + S(BC) - S(C) - S(ABC) \end{aligned} \tag{3.22}$$

which illustrates that the conditional mutual entropies are always non-negative as a consequence of the strong subadditivity property of quantum entropies. The entropy in the center of the diagram is a *ternary* mutual entropy, defined as

$$\begin{aligned} S(A{:}B{:}C) &= S(A{:}B) - S(A{:}B|C) \\ &= S(A) + S(B) + S(C) - S(AB) \\ &\quad -S(AC) - S(BC) + S(ABC) \end{aligned} \tag{3.23}$$

and corresponds to the entropy shared by the three subsystems A, B, and C. Note that for any tripartite system in a pure state, we have $S(AB) = S(C)$, $S(AC) = S(B)$, and $S(BC) = S(A)$, so that the *ternary* mutual entropy vanishes. More generally, for a multipartite system, relations between quantum entropies can be written which parallel the classical relations and have the same intuitive interpretation.

As an illustration, let us consider a tripartite system ABC in a Greenberger-Horne-Zeilinger (GHZ) state (which will become crucial in the quantum measurement process), described by the wave function

$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle) \; . \tag{3.24}$$

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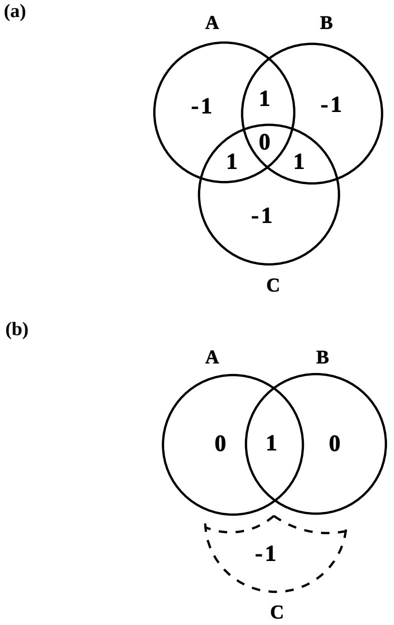
[5] Note that the spectrum of $\rho_{A|B}$ and $\rho_{B|A}$ is invariant under local transformations of the form $U_A \otimes U_B$.

[6] Peres' criterion of separability [23] is that none of the eigenvalues of the partial transpose of ρ_{AB} is negative. For the Werner state, three eigenvalues are equal to $(1+x)/4$ and the fourth one is equal to $(1-3x)/4$. This lowest eigenvalue is non-negative if $x \leq 1/3$. Thus, expressing that these eigenvalues are non-negative is simply equivalent to expressing that the eigenvalues of $\rho_{A|B}$ do not exceed one.

FIG. 3. (a) Ternary entropy diagram for a GHZ state (an "EPR-triplet"). (b) Entropy diagram for subsystem AB, unconditional on C. The entropy of C conditional on AB is negative, and compensates the positive entropy of AB unconditional on C.



As it is a pure state, its quantum entropy is $S(ABC)=0$. When tracing over any degree of freedom (for instance the one associated with C), we obtain

$$\rho_{AB}=\frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow|+|\downarrow\downarrow\rangle\langle\downarrow\downarrow|) \tag{3.25}$$

corresponding to a classically correlated system of type II (see Fig. 1b). We thus find $S(A)=S(B)=S(C)=S(AB)=S(AC)=S(BC)=1$, allowing us to fill in the entropy diagram[7] for the GHZ state in Fig. 3a. The important feature of the GHZ state is that it entails quantum entanglement between *any* part (e.g., C) and the rest of the system (AB). Even more important, ignoring (that is, tracing over) a part of it (C) creates *classical* correlation between the two remaining parts (A and B), as shown in Fig. 3b. In other words, the subsystem AB *unconditional* on C, i.e., without considering the state of C, is indistinguishable from a type II system. This property is *central* to the understanding of the quantum measurement process, and will be emphasized throughout the following section. It is generalized without difficulty to the case of an "EPR-nplet":

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle+|\downarrow\downarrow\cdots\downarrow\rangle)\,. \tag{3.26}$$

Ignoring (tracing over) any degree of freedom creates classical correlations between all the remaining degrees of freedom.

We can see now how an EPR entangled system (an EPR pair) plays a special role in quantum mechanics. The correlation between the elements of the pair [described by the mutual entropy $S(A{:}B)$] goes beyond anything classically achievable ("super-correlation"). A classical approach to understanding the correlations suggests that measuring half of an EPR pair *immediately* affects the other half, which may be arbitrarily far away. Classical thinking of this sort applied to an EPR pair is misleading, however. Indeed, a careful investigation of the information flow in EPR pair experiments reveals that causality is never violated. In Ref. [1] we suggest that EPR pairs are better understood in terms of qubit–antiqubit pairs, where the qubit (antiqubit) carries plus (minus) one bit of information, and antiqubits are interpreted as qubits traveling *backwards* in time[8]. In anticipation of the discussion in the following section, let us mention (as advertised earlier) that the von Neumann measurement [see Eq. (2.8)] creates just such EPR entanglement (not classical correlation) between the quantum system and the measurement device. The key realization will be that the quantum von Neumann entropy rather than Shannon-Boltzmann-Gibbs entropy is in fact the *physical* entropy [1, 2]. This explains the observation that entropy is created in the measurement of the spin of, say, an electron, in spite of the fact that the von Neumann entropy is zero for a pure state, independently of the choice of basis. As we outline below, the apparent entropy created in a spin measurement (if the spin is not aligned with the measurement axis) is actually the quantum entropy of *part* of an entangled system, and is cancelled by the negative conditional entropy of the (non-observed) remainder.

IV. MEASUREMENT PROCESS

A. Second stage: observation

We have now prepared the ground to understand von Neumann's second stage. The crucial observation was

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[7] The negative conditional entropies in this diagram betray that this state is purely quantum, unobtainable in classical physics. As mentioned earlier, the fact that the *ternary* mutual entropy $S(A{:}B{:}C)$ is zero is generic of the description of any three-body system in a pure state [it follows from the constraint $S(ABC)=0$, i.e., that ABC has been formed by applying a unitary transformation on a pure state].

[8] The term *qubit* denotes the quantum unit of information, which is the quantum analog to the classical unit of information), see, e.g. [24].

touched upon briefly above: von Neumann entanglement (2.8) creates *super-correlations* (a type III EPR-entangled state) between Q (measured quantum system) and A (ancilla), rather than correlations. The system QA thus created is inherently quantum, and cannot reveal any classical information. To obtain the latter, we need to create classical correlations between *part* of the EPR-pair QA and *another* ancilla A', i.e., we need to observe the quantum observer. No new ingredients are needed for this. Rather, we simply allow the EPR entangled system QA to come into contact with a system A', building the system QAA'. Subsequently, we apply a unitary transformation with an interaction Hamiltonian of the type (2.5), only that now it is defined on the combined Hilbert space of QA and A'. Clearly, this is just a repetition of the first stage, but now leading to a GHZ-like state[9]

$$|QAA'\rangle = |x,x,x\rangle + |y,y,y\rangle \ . \tag{4.1}$$

All operations have been unitary, and QAA' is described by the pure state

$$\rho_{QAA'} = |QAA'\rangle\langle QAA'| \ . \tag{4.2}$$

Experimentally, however, we are *only* interested in the correlations between A and A', and *not* in correlations between A and Q (which are unobservable anyway). Luckily, there is no obstacle to obtaining such classical (type II) correlations now (unlike in the case where only two particles were quantum entangled). Indeed, it is now immediately obvious that when ignoring the quantum state Q *itself*, as paradoxically as it may appear at first sight, A and A' find themselves classically correlated and in a *mixed* state:

$$\rho_{AA'} = \mathrm{Tr}_Q(\rho_{QAA'}) = |x,x\rangle\langle x,x| + |y,y\rangle\langle y,y| \ . \tag{4.3}$$

We will show that ignoring Q turns out to be unavoidable when measuring Q. This is the basic operation (ignoring part of an "EPR-nplet") that was alluded to in the previous section, and which we will encounter again below.

In general, for the measurement of any quantum system in an N-dimensional discrete Hilbert space we obtain after tracing over Q

$$\rho_{AA'} = \sum_{i=1}^{N} p_i \, |ii\rangle\langle ii| \tag{4.4}$$

where the p_i are the probabilities to find A (or A') in one of its eigenstates $|i\rangle$. This completes the second stage of the quantum measurement. A state was formed (AA') which *appears* to be mixed,

$$S(AA') > 0 \ , \tag{4.5}$$

while A, A' and Q were pure to begin with. Yet, this mixed state is quantum entangled with Q, which carries negative conditional entropy

$$S(Q|AA') < 0 \tag{4.6}$$

such that the combined system QAA' is still pure:

$$S(QAA') = S(AA') + S(Q|AA') = 0 \ . \tag{4.7}$$

Clearly therefore, a transition from a pure state to a mixed state (for the entire isolated system QAA') did *not* take place, whereas the quantum probabilities in the mixed state AA' correspond *precisely* to the square of the amplitudes of quantum mechanical measurement (see Section V). Quantum probabilities arise in unitary time development, thanks to the negative entropy of the "unobserved" quantum system Q.

Let us emphasize now the fact that this view of measurement implies that conceptually *three* rather than just two systems must be involved. The "observation" of the measurement is possible only when a third system A' (a quantum particle or set of particles with a Hilbert space dimension at least equal to the dimension of the Hilbert space of Q) interacts with A (the ancilla which "measured" Q through von Neumann entanglement). Indeed, the classical intuition of measurement is built upon *correlations*, which can only emerge in the presence of a *third* system A'. The fact that A' need not be a microscopic object is an issue which will become important when we will be concerned with the *amplification* of the measurement. But, conceptually speaking, it is enough to say that A' is a particle that "observes" the measurement made by A on Q. Because classical observers are necessarily made out of a macroscopic number of particles, it is *in practice* necessary to have a large number of correlated particles $A', A'', \cdots$ in order to achieve a macroscopic measurement. However, this is completely arbitrary: we may say that a measurement has been performed as long as the result is recorded on any kind of storage device[10], in which case the size of $A', A'', \cdots$ simply depends on the number of particles in the measurement apparatus. As a matter of fact, just *one* particle living in the same Hilbert space as Q and A is enough to complete a conceptual measurement, so that the description of the system QAA' is enough to completely model quantum measurement.

Our model does therefore not fall in the class of environment-induced decoherence models, simply because information is not lost to an environment. We have a quantum state Q and an ancilla A (which may be composed of very few degrees of freedom, and does not have to be "large"). We suggest that a measurement simply

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[9] We dispense with normalizations.

[10] This is the content of the so-called "psychophysical parallelism" hypothesis, that a measurement is achieved whether or not a conscious observer is involved [6].

implies ignoring the quantum system Q, which forces the ancilla A to appear in a mixed state. Our model does *not* predict the quantum system Q to be classically correlated with the ancilla A after the measurement, the cornerstone of standard environment-induced decoherence models. Rather, we argue that the classical correlations that emerge from the measurement (by tracing over Q) concern the internal degrees of freedom of A only. The ancilla is therefore "self-consistent", since arbitrarily dividing A into two halves always provides two classically correlated subsystems. In other words, A is never correlated with Q; correlations only appear inside A. Thus, our description appears to be more fundamental, as it can account for a measurement situation where the degrees of freedom of A are few and totally controllable (they are not traced over). In contrast, environment-induced decoherence models cannot explain the appearance of mixed states in such systems (see, e.g., [14]). Of course, our model does not preclude a more complex situation where a macroscopic uncontrollable environment is coupled with Q and A, but we believe such an environment is not conceptually necessary to interpret a measurement. The apparent irreversibility (creation of entropy) is traced to the "hidden" negative entropy inside the measured quantum system itself, not to the large environment.

As will be emphasized in Section V, the illusion of a wave-function collapse can be understood by considering consecutive measurements. A subsequent observation of Q (which is now part of an entangled system QAA') with another ancilla, say BB', will result in BB' showing the *same* internal correlations as AA'. More importantly, the second ancilla will be 100% correlated with the first, implying that it reflects the same exact outcome. This leaves the observer with the illusion that one definite outcome was recorded by the first ancilla and that any subsequent measurement simply confirms that Q is in that state. In other words, it appears as if the first measurement projected the quantum state onto an eigenstate, as reflected by any subsequent measurement. Yet, the only effect of the measurement on the quantum state is entanglement with the devices, and all amplitudes of the quantum system are unchanged. Partial observation of the entangled state leads to all the devices being 100% correlated.

B. Amplification and reversibility

As mentioned above, inducing classical correlations between the quantum variables A and A' does not lead to a macroscopically observable pointer. Rather, the basic unitary operation

$$(QA) + A' \xrightarrow{U} QAA' \tag{4.8}$$

must be "repeated" $O(10^{23})$ times until a macroscopic number of quantum particles $(A', A'', \cdots)$ are correlated with A, such that the result can be observed and *recorded*. The quantum state of the joint system $QAA'A'' \cdots$ is akin to an entangled EPR-nplet with vanishing entropy. An experimental setup allows the observation of the correlations between A and $A'A'' \cdots$ *unconditional* on Q (ignoring the quantum state itself), and results in all of the 10^{23} particles reproducing (being classically correlated with) the quantum state of A. This process is usually called the amplification, or "classicization", of the quantum state A. The two-stage measurement process including entanglement and amplification is pictured in Fig. 4.

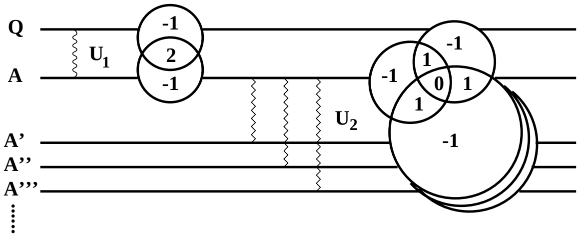


FIG. 4. Diagrammatic representation of the two-stage unitary measurement. EPR-entanglement between measured quantum system Q and ancilla A (first stage, U_1) and entanglement between QA and macroscopic system $A'A'' \cdots$ (second stage, observation U_2). The macroscopic ancilla $AA'A'' \cdots$ *unconditional* on Q is a mixed state describing classical correlation. However, Q and $AA'A'' \cdots$ still form an EPR-pair.

Before turning to the question of reversibility, let us stress the fact that the creation of entropy (in a subsystem) depends on the initial state of Q with respect to the observable under consideration. The fact that an *arbitrary* state cannot be duplicated (or cloned) plays a crucial role in the amplification process: the quantum noncloning theorem [11] states that it is possible to amplify a quantum state (e.g. the state of A) *only* if it belongs to a set of orthogonal states. More precisely, when a quantum system Q is allowed to interact with an ancilla A in order to measure an observable O_A, the eigenstates $|a\rangle$ of O_A define the set of orthogonal states that can be amplified (and which lead to a macroscopic device that reflects the microscopic state). An attempt at amplifying an *arbitrary* quantum state *will* generate entanglement between the particles constituting the macroscopic object. This entanglement then is responsible for the generation of randomness in the outcome. Accordingly, subsystem $(AA'A'' \cdots)$ carries positive *unconditional* entropy, while the unobserved Q (which is traced over) carries the commensurate negative *conditional* entropy to allow for the zero entropy pure state of the entire entangled system $QAA'A'' \cdots$.

Let us close this section by stressing that, while quantum measurement is conceptually reversible, its irreversible appearance has the same roots as irreversibility in classical mechanics, as suggested earlier by Peres [4]. As explained previously, the amplification consists in repeating the basic von Neumann measurement a large number of times, until a macroscopic number of quantum particles are correlated with A. The *whole* (isolated) system is in a *pure* entangled state, but ignoring (tracing over) Q makes the rest of the system appear classically correlated. Yet, *no irreversible process* takes place. Randomness [the probabilities p_n in (4.4)] is generated because A already *appears* to be random if one fails to take into account Q. This is the measurement analogue of the random orientation of half of an EPR pair in an otherwise fully determined ($S = 0$) system. Nothing new happens by introducing correlations between A and a macroscopic number of quantum particles ($A'A'' \cdots$). However, *reversing* the "observation" operation [applying a sequence of inverse unitary transformations of the type (4.8)] turns out to be exceedingly difficult in practice. Indeed, one would have to reverse *every one* of the $O(10^{23})$ unitary operations that introduced the correlations between the macroscopic set of particles. While this is possible in principle, it is practically not so because missing a single particle that was involved in the measurement would result in the incorrect unitary (inverse) transformation, thus failing to restore the initial quantum state. The root for the practical irreversibility is thus the same for the quantum measurement as for the physics of macroscopic classical systems. The temporal development is irreversible only because of the practical impossibility to control a macroscopic number of initial conditions, while the microscopic interactions are all reversible.

As a consequence, we see that only those quantum measurements can be reversed for which the ancilla A is *not* correlated with a macroscopic number of particles, i.e., when A is not explicitly observed by a macroscopic observer. However, the reversibility of the first stage of the measurement, the quantum entanglement, can, and *has been*, achieved. Common lore of double-slit experiments holds that just providing the *possibility* of performing a measurement (providing the opportunity to obtain "which-path" information, for example) is irreversible. As illustrated by the so-called "quantum-eraser" experiments, this is incorrect [25]. Indeed, providing the possibility of observation (rather than measurement itself) is, according to the unitary quantum measurement theory outlined here, just the von Neumann measurement (the first stage, or EPR entanglement), and is therefore completely reversible. In Appendix A, we analyse the quantum eraser setup within our framework.

V. INCOMPATIBLE MEASUREMENTS AND UNCERTAINTY RELATIONS

We will now show that the uncertainty principle which characterizes the measurement of two incompatible observables arises naturally from our unitary description of the measurement process. We also derive a new bound for the entropic uncertainty relation for consecutive measurements which is stronger than the one in the literature to date.

Let us perform two consecutive measurements on the quantum system Q. First, we measure the observable O_A by allowing Q to interact with a (first) ancilla A. (The amplification stage of the measurement is ignored here for the sake of simplicity). Subsequently, we let the system Q interact with an ancilla B in order to measure observable O_B. For illustrative purposes, we assume that Q is a discrete system which is initially described by the state vector

$$|Q\rangle = \sum_{i=1}^{N} \alpha_i |a_i\rangle \tag{5.1}$$

where $|a_i\rangle$ are the eigenstates of O_A and N is the dimension of the Hilbert space associated with Q (or A, or B). The unitary transformation associated with the measurement of O_A creates an entangled state for the joint system QA

$$|QA\rangle = \sum_{i=1}^{N} \alpha_i |a_i, i\rangle \tag{5.2}$$

where $|i\rangle$ are the basis states of A, which label the different outcomes of the first measurement. In other words, if Q is in state $|a_i\rangle$, the ancilla A ends up in state $|i\rangle$. As explained previously, if Q is initially *not* in one of the eigenstates of O_A, QA will be entangled. Of course, $S(QA) = 0$, since it evolved unitarily from the pure state $|Q, 0\rangle$. The marginal density matrix of A is obtained by tracing the density matrix $\rho_{QA} = |QA\rangle\langle QA|$ over Q, yielding

$$\rho_A = \sum_i |\alpha_i|^2 |i\rangle\langle i| \ . \tag{5.3}$$

Consequently, the quantum entropy of A is given by

$$S(A) = H[p_i] \tag{5.4}$$

where $H[p_i]$ denotes the classical (Shannon) entropy

$$H[p_i] = -\sum_i p_i \log p_i \tag{5.5}$$

associated with the probability distribution $p_i = |\alpha_i|^2$. This is in perfect agreement with the standard description of a measurement, which states that the outcome i occurs with a probability $p_i = |\alpha_i|^2 = |\langle a_i|Q\rangle|^2$, i.e., it is simply the square of the quantum amplitude α_i. Remarkably thus, the physical (von Neumann) entropy of A reduces *precisely* to the Shannon entropy for the outcome of the measurement, which is the one predicted by standard quantum mechanics. Yet, since A is entangled with Q, the physical entropy of the combined system remains zero.

We now consider the measurement of the second observable O_B, by letting Q interact with B. First, we define the unitary operator U which transforms the eigenstates $|a_i\rangle$ of O_A into the eigenstates $|b_j\rangle$ of O_B: its matrix elements are[11]

$$U_{ij} = \langle b_j | a_i \rangle \; . \tag{5.6}$$

Obviously, if O_A and O_B commute, U is the identity matrix. Expressing $|Q\rangle$ in the $|b_j\rangle$ basis and entangling it with B in order to measure O_B, we obtain the final state of the system

$$|QAB\rangle = \sum_{i,j=1}^{N} \alpha_i U_{ij} |b_j, i, j\rangle \tag{5.7}$$

where $|j\rangle$ are the basis states of B (again, this means that if B is in state j then Q was initially in b_j). This is also an entangled state, with zero entropy $[S(QAB) = 0]$ since it was obtained by evolving a pure state using two unitary transformations. The marginal density matrix describing AB (ignoring the system Q) is given by

$$\rho_{AB} = \sum_{i,i',j} \alpha_i \alpha^*_{i'} U_{ij} U^*_{i'j} |i, j\rangle\langle i', j| \; . \tag{5.8}$$

Note that ρ_{AB} *cannot* be diagonalized by applying a change of variable of the product form $(U_A \otimes U_B)$, except in the case where O_A and O_B commute. The marginal density matrices for A and B are given by

$$\rho_A = \sum_i |\alpha_i|^2 |i\rangle\langle i| \; , \tag{5.9}$$

$$\rho_B = \sum_{i,j} |\alpha_i|^2 |U_{ij}|^2 |j\rangle\langle j| \; . \tag{5.10}$$

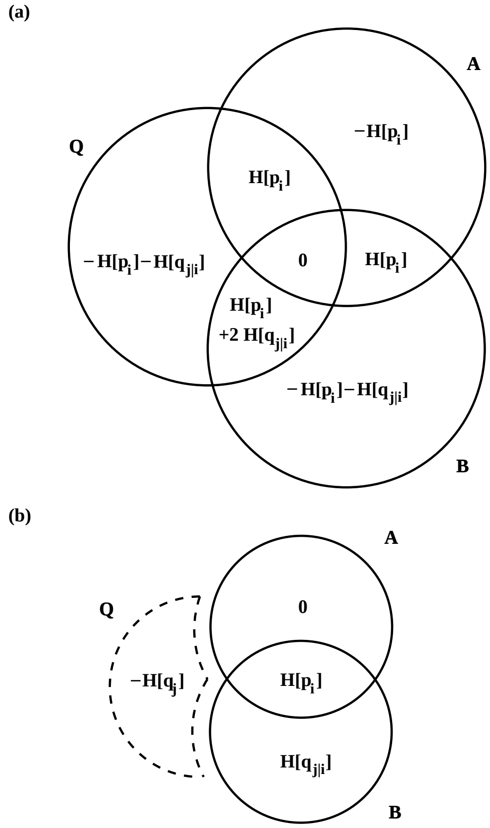
The quantum entropies of A and B then read

$$S(A) = H[p_i] \qquad \text{with } p_i = |\alpha_i|^2 \; , \tag{5.11}$$

$$S(B) = H[q_j] \qquad \text{with } q_j = \sum_i p_i q_{j|i} \; . \tag{5.12}$$

where $q_{j|i} = |U_{ij}|^2$ and $H[q_j]$ is the classical (Shannon) entropy associated with the probability distribution q_j. Here, $q_{j|i}$ can be understood as the *conditional* probability to obtain the outcome j for the second measurement, after having obtained outcome i for the first one.

FIG. 5. (a) Ternary entropy diagram for the system QAB (quantum system Q, and ancillae A and B). (b) Entropy diagram of the system AB unconditional on Q, describing the sequential measurement of O_A and O_B.



Remarkably, the entropy of the second measurement $H[q_j]$ is completely compatible with the standard assumption of a collapse of the wave function in the first measurement. Indeed, it corresponds exactly to what would be predicted in conventional quantum mechanics, by assuming that the wave function was projected on $|a_i\rangle$ with a probability $p_i = |\alpha_i|^2$ after the first measurement, and interpreting $|U_{ij}|^2$ as the probability of measuring j on an eigenstate $|a_i\rangle$ of the first observable. This reveals how the standard assumption of wave function collapse in measurement can be *operationally* correct, although we show here that it is not the actual physical process. Note that the first measurement can be viewed as inducing a "loss of coherence", as the second measurement yields $q_j = \sum_i |\alpha_i U_{ij}|^2$ rather than $q_j = |\sum_i \alpha_i U_{ij}|^2 = |\langle b_j|Q\rangle|^2$, as would be the case if there was no first measurement. For the combined system QAB on the other hand, there is of course no loss of coherence.

The entropy diagram corresponding to the state QAB is shown in Figure 5a. The entropy of A (resulting from the first measurement) is $S(A) = H[p_i]$, whereas the entropy of B (resulting from the second measurement) is

$$S(B) = H[q_j] = H[p_i] + H[q_{j|i}] \; , \tag{5.13}$$

where we defined the (classical) *conditional* entropy

$$H[q_{j|i}] = -\sum_{i,j} p_i q_{j|i} \log q_{j|i} \; . \tag{5.14}$$

This last quantity represents the *additional* amount of entropy that appears due to the second measurement. Figure 5b depicts the apparent entropy diagram of AB unconditional on Q, illustrating the basic equation

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[11] This unitary operation is unique up to a permutation of eigenstates which is unimportant in this discussion.

$$S(A) + S(B|A) = H[p_i] + H[q_{j|i}] = H[q_j] \tag{5.15}$$

relating the entropy of the first and the second measurement. Note that, despite the asymmetry between A and B (O_A is measured first), Eq. (5.15) can be rewritten in symmetric form

$$S(A) + S(B) \geq H[q_j] \tag{5.16}$$

since the mutual entropy $S(A{:}B)$ is always positive. Equation (5.15) plays the role of an uncertainty relation for entropies, expressing the fact the the sum of the entropies resulting from the measurement of O_A and O_B is constant. If we were to try to reduce the entropy associated with one of them, then the other entropy would increase. In order to have a genuine "entropic uncertainty relation" for consecutive measurements, independent of the initial state of Q, it is necessary to minimize the right-hand side of Eq. (5.15) over $|Q\rangle$ (i.e., over the α_i's). The convexity of Shannon entropy implies that $H[q_j]$ is minimized in the case where the p_i distribution is maximally peaked, that is, when the initial state of Q is an eigenstate $|a_i\rangle$ of the first observable. In this case, $S(A{:}B) = H[p_i] = 0$, and therefore, assuming $|Q\rangle = |a_i\rangle$ (for instance) yields

$$S(A) + S(B) \geq H[q_{j|i}]_{i \text{ fixed}} \equiv -\sum_j q_{j|i} \log q_{j|i} \tag{5.17}$$

Then, minimizing over i, we obtain the entropic uncertainty relation

$$S(A) + S(B) \geq \min_i H[q_{j|i}]_{i \text{ fixed}} = \min_i H[|U_{ij}|^2]_{i \text{ fixed}} \tag{5.18}$$

Physically, this means that the sum of the entropies is bounded from below by the Shannon entropy corresponding to the expansion of an eigenstate of O_A into the basis of eigenstates of O_B (more precisely, the eigenstate which minimizes the Shannon entropy). Note that our entropic uncertainty relation (5.18) is stronger than the Deutsch-Kraus exclusion principle [26, 27, 28], which states that

$$S(A) + S(B) \geq -\log c \tag{5.19}$$

where $c = \max_{i,j} |U_{ij}|^2$. Indeed, it is easy to see that $\min_i H[|U_{ij}|^2]_{i \text{ fixed}} \geq -\log c$.

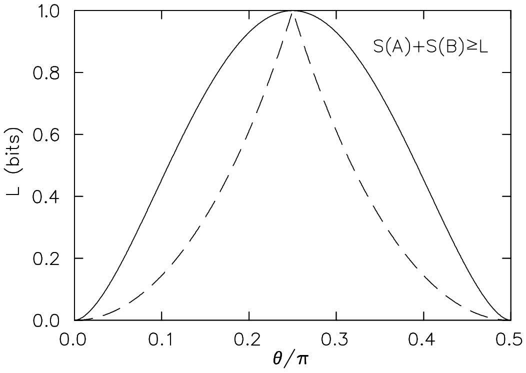
In the case of complementary observables (i.e., if the distribution of O_A values is uniform for any eigenstate of O_B and vice versa), one obtains the simple entropic uncertainty relation [27, 29]

$$S(A) + S(B) \geq \log N \tag{5.20}$$

where N is the dimension of the Hilbert space, as expected. This bound just corresponds to the situation where the conditional entropy $S(Q|AB)$ takes on the largest negative value compatible with the dimension of the Hilbert space of Q. This is for instance the case if one measures any two spin-projections of a spin-1/2 particle. In this case, we obtain

$$S(\sigma_x) + S(\sigma_y) \geq 1\ . \tag{5.21}$$

FIG. 6. Lower bound for the entropic uncertainty relation in a spin-1/2 Hilbert space. The solid line represents our bound [Eq. (5.23)], while the dashed line stands for the Deutsch-Kraus bound [Eq. (5.22)], for θ between 0 and $\pi/2$.



For the case of two commuting observables ($[O_A, O_B] = 0$), we find $U_{ij} = \delta_{i,j}$ and therefore $S(A) + S(B) \geq 0$, reflecting that they can be measured simultaneously with arbitrarily high accuracy. In situations that are intermediate between compatible and complementary (maximally incompatible) observables, our bound is demonstrably more constraining than the one of Deutsch and Kraus. Let us show this for the simple case of a two-dimensional Hilbert space.

For a general 2×2 unitary matrix U_{ij}, with $|U_{11}|^2 = |U_{22}|^2 = \cos^2\theta$, $|U_{12}|^2 = |U_{21}|^2 = \sin^2\theta$, and θ an angle parameter, the Deutsch-Kraus uncertainty relation is

$$S(A) + S(B) \geq -\log \max\left\{\cos^2\theta, \sin^2\theta\right\}\ , \tag{5.22}$$

whereas we find

$$S(A) + S(B) \geq H\left[\cos^2\theta, \sin^2\theta\right]\ . \tag{5.23}$$

In Fig. 6, we compare the right-hand sides of Eqs. (5.22) and (5.23), illustrating that the bounds are equal only for completely compatible ($\theta = 0, \pi/2$) or maximally incompatible ($\theta = \pi/4$) observables.

VI. INTERPRETATION

In this section we comment on the implications of unitary quantum measurement and the concept of quantum entanglement for the foundations and the interpretation of quantum mechanics.

The inability to consistently describe the measurement process in quantum mechanics–the quantum measurement paradox–has seriously discredited the foundations of a theory that otherwise describes the microscopic world succinctly, effortlessly, and correctly. The questions that we would like to address anew here concern the relation between quantum and classical concepts, the Schrödinger "cat paradox", as well as the interpretation of the complementarity principle.

In standard quantum mechanics, the criterion to decide whether a classical or a quantum picture is more adequate generally involves comparing a representative unit of action of the system under consideration $S_{\rm typ}$ with the unit $\hbar$. Such a criterion suggests that any macroscopic system that fulfills $S_{\rm typ} \gg \hbar$ behaves classically. Yet, the present paper proposes that EPR-entangled systems, whether microscopic or macroscopic, are fundamentally quantum and can in *no limit* be understood classically. We would like to suggest here that a degree of freedom *appears* classical if it is composed of many $[O(10^{23})]$ classically correlated internal variables. This occurs precisely when part of an entire isolated system which is in a pure quantum state is ignored (i.e., unobserved and traced over). Note that tracing over just *one* degree of freedom that is entangled is enough to promote the classical appearance! Tracing over an unobserved degree of freedom is *not* a physical process, and is thus not described by any time evolution. Rather, a quantum measurement forces the observation of correlations *unconditional* on part of a (quantum inseparable) system. Thus, any classical degree of freedom has a "classical appearance" only because it is part of a larger quantum inseparable system in a pure state.

Let us consider this in more detail, as it suggests a very simple and satisfying explanation for the Schrödinger cat paradox. In this, perhaps the most well-known and most puzzling of all *gedankenexperimente*, the first stage of the measurement concerns a decaying atom and its emitted particle (say, a photon). Let us assume, as is usual, that the wavefunction (after some time) is a superposition of an "excited" atom $A^\star$ and the vacuum, and a decayed atom A with one photon:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\big(|A^\star,0\rangle + |A,1\rangle\big)\,, \tag{6.1}$$

i.e., both atom and photon form an entangled state with vanishing overall entropy. Then, in the second stage of the measurement, the $O(10^{23})$ atoms forming the cat interact with the photon, forming an EPR-nplet for the entire quantum state – of course still a pure state. If we simplify the problem by assuming that the cat's quantum variable is dichotomic, with live and dead cat eigenstates $|L\rangle$ and $|D\rangle$, the wave function becomes

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}\bigg(|A^\star,0,L\rangle + |A,1,D\rangle\bigg)\,. \tag{6.2}$$

Tracing over the initial atom (the experiment after all involves monitoring the cat, not the atom), one obtains a mixed state where all the 10^{23} atoms are *correlated* with the emitted particle, i.e., they are arranged in such a way that the cat is either dead or alive (with probabilities 1/2):

$$\rho_{\gamma,{\rm cat}} = \frac{1}{2}\bigg(|0,L\rangle\langle 0,L| + |1,D\rangle\langle 1,D|\bigg)\,. \tag{6.3}$$

This macroscopic system has an entropy of 1 bit, that is, randomness has been created. More importantly, the density matrix is equal to that of a statistical ensemble prepared with equal numbers of dead and living cats, making both situations (the experiment and the preparation) physically *indistinguishable*. The randomness created in the cat-γ subsystem is compensated by a conditional entropy of –1 bit for the decaying atom. Since the entire system has vanishing entropy, it is still completely determined. Moreover, no such thing as a collapse of the cat wave function happens when the box is opened to an observer; what happens is simply that now all the atoms of the observer become also entangled with those of the cat:

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\bigg(|A^\star,0,L,l\rangle + |A,1,D,d\rangle\bigg)\,. \tag{6.4}$$

where we introduced the dichotomic observer states $|l\rangle$ and $|d\rangle$ describing the observation of the live or dead cat. The corresponding marginal density matrix is

$$\rho_{\gamma,{\rm cat},{\rm obs}} = \frac{1}{2}\bigg(|0,L,l\rangle\langle 0,L,l| + |1,D,d\rangle\langle 1,D,d|\bigg)\,. \tag{6.5}$$

The observer notices that the cat is either dead or alive, and thus the observer's own state becomes classically correlated with that of the cat, although, in reality, the entire system (including the atom, the γ, the cat, and the observer) is in a *pure entangled* state. It is *practically* impossible, although not in principle, to undo this observation, i.e., to resuscitate the cat, or, more precisely, to come back to the initial decaying atom, with a living cat and an ignorant observer

$$|\Psi_2\rangle \xrightarrow{U_2^{-1}} |\Psi_1\rangle \xrightarrow{U_1^{-1}} |\Psi_0\rangle\,, \tag{6.6}$$

since it requires to enact the inverse unitary transformations on all the atoms forming the observer and the cat. This irreversibility is completely equivalent to the irreversibility in classical mechanics. Indeed, classically, to reverse the microscopic time evolution, it is necessary to invert the velocity of all the particles, the practical impossibility of which gives a macroscopic irreversible aspect to time evolution. In quantum mechanics, it is necessary to undo any unitary evolution associated with all interactions that particles have undergone, so that reversibility is practically impossible if a macroscopic number of particles have been involved. We are led to conclude that irreversibility is *not* an inherent feature of quantum mechanics.

Finally, the present approach sheds light on the origins of the complementarity principle, or wave-particle duality. On the one hand, we see that the wave function *completely* describes a quantum state, a fact eloquently argued for by Bohr. On the other hand, we cannot escape the appearance of randomness in quantum measurement. These facts were interpreted by Bohr to be "complementary" to each other, much as the wave nature of quantum objects was viewed as "complementary" to its particle nature. Our identification of von Neumann entropy as the real, physical, entropy of a system corroborates that the quantum wave function does indeed provide a complete description of the quantum state, since the von Neumann entropy of a pure state is zero. Yet, we find that randomness is not an essential cornerstone of quantum measurement, but rather an illusion created by it. Thus, we are led to conclude that complementarity is a working concept, but has no ontological basis as a principle. The same appears to be true for the wave-particle duality. On the one hand we agree that quantum systems, due to the superposition principle, are wave-like in nature. This is inherent in the "completeness postulate of the density matrix" (see, e.g., [10]), which implies that two systems prepared in the same density matrix, but by making different mixtures of pure states, are completely *indistinguishable*. On the other hand, the particle aspect of a quantum object emerges simply from the measurement process, when a wavefunction interacting with a measurement device appears as a mixed state. Thus, as we unmask the particle-like behavior of quantum systems to be an *illusion* created by the incomplete observation of a quantum (entangled) system with a macroscopic number of degrees of freedom, we are led to conclude once more that the wave-nature of quantum systems is *fundamental*, and that there is no particle-wave duality, only an apparent one.

VII. CONCLUSION

In conclusion, we are able to reconcile unitary evolution of quantum states and the apparent creation of randomness in a minimal model of the measurement process. This is achieved via the introduction of an elementary quantum measurement process (the EPR entanglement) in which entropy is conserved by balancing randomness with negative entropy. We show how the usual probabilistic results of quantum mechanics arise naturally in this description, paving the way for a fully consistent description of quantum mechanics in which the measurement device is *not* accorded a privileged role. This description does not require the concept of wave function collapse or the presence of a macroscopic environment in order to predict the results of quantum experiments, thereby removing the special status of quantum mechanics as far as irreversibility is concerned. In addition, our analysis shows that, in spite of its appearance, any classical system is in fact an entity which is part of a larger quantum system. We believe this answers the question about the location of the frontier between the quantum and the classical world, with respect to measurement. We answer that there is no classical world, only the classical appearance of part of a quantum world. This view is especially satisfying as measurement, bereft of its special status outside of quantum mechanics (which it had been accorded to by the Copenhagen interpretation) and unencumbered by external notions such as consciousness (as advocated by von Neumann) is now part of a consistent theory defined without recourse to classical notions which, after all, should appear as a limit of a quantum theory only.

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APPENDIX A: STANDARD QUANTUM EXPERIMENTS

In this appendix we apply our quantum measurement theory to standard experiments, in order to illustrate how the usual quantum probabilistic results emerge in a unitary treatment.

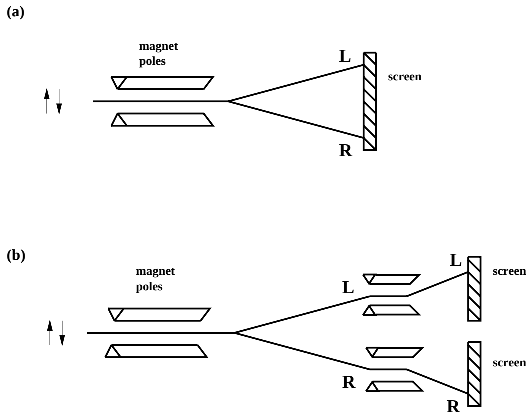
1. Stern-Gerlach experiment

In the Stern-Gerlach experiment, a beam of atoms is guided through an inhomogeneous magnetic field B_z normal to the direction of motion of the atoms (see Fig. 7a). In this field, the atoms experience a force deflecting them out of the beam, depending on the orientation of their magnetic moments with respect to the magnetic field axis. The beams are collected a distance away on a screen. Let us assume here for simplicity that the magnetic moments of the atoms take on only two different values ($s = 1/2$), and define σ_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$. If the incident beam consists out of atoms prepared in a σ_x (say) eigenstate, the initial state is a quantum superposition

$$|\Psi_{\rm beam}\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle + |\downarrow\rangle\big)\;. \tag{A1}$$

The auxiliary variable, or ancilla, is in this case a spatial location, say left or right (L, R). Applying the magnetic field then completes the von Neumann measurement

FIG. 7. (a) Setup of the Stern-Gerlach experiment. (b) "Consistency" requirement for two sequential Stern-Gerlach experiments illustrating the appearance of classical correlation.



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow, L\rangle + |\downarrow, R\rangle\big)\ . \tag{A2}$$

Through this operation, the different spin-orientations have been "tagged" (the ↑ spin is tagged with a left location, and conversely), but it is incorrect to assume that spin-orientations and locations are now *correlated*. Much more than that, they are *entangled*: locations and spin-orientations form EPR pairs. The second stage of measurement (amplification) occurs on the screen. Collecting the particles ignores the spin-orientation entirely such that the particles of the screen become classically correlated with the location variable, forming a type II classically correlated system carrying one bit of entropy. Let us emphasize here that the measurement of the location variable (L, R) does *not* allow us to infer the spin orientation of the atom. Thus, even though the particle beam was deflected in the z-direction (as if the beam was composed of atoms with magnetic moments quantized in the z-direction), such a classical description is misleading.

Denoting as usual the system (atom) with Q, the ancilla (location) with A and the screen with A' (with eigenstates $|l\rangle$ and $|r\rangle$), we obtain

$$\rho_{AA'} = \mathrm{Tr}_Q(\rho_{QAA'}) = \frac{1}{2}\bigg(|L,l\rangle\langle L,l| + |R,r\rangle\langle R,r|\bigg) \tag{A3}$$

which is the standard result: the spot on the screen reflects the $L-R$ variable (classical correlation). Yet, the entropy of the combined system QAA' has not changed, still being zero. The randomness in the measurement result (the bit of entropy in the AA' system) is cancelled by the negative entropy of the unobserved quantum state Q,

$$S(Q|AA') = -1\ . \tag{A4}$$

It is important to observe that the randomness which may appear in the measurement of the position (collecting the particles on a screen or a detector) does *not* occur because there were unknown internal degrees of freedom, which along with the wave function, would be needed to completely describe the particle (cf. hidden-variable theory). The wave function *entirely* defines the state (it is indeed of zero entropy).

It is well-known that if a second magnetic field gradient is used in order to perform a second Stern-Gerlach measurement (foregoing the collection on the screen) as depicted in Fig. 7b, one obtains two correlated variables: the position x after the first, and y after the second field gradient. The standard interpretation is that, once the wave function has been projected (via the first field gradient), only positive (negative) spin-projection particles are left in the $L(R)$ beam, so that the second measurement is incapable of splitting the beam again. This is a basic requirement for consecutive measurements of the same observable on a quantum system. In reality, this is nothing else than the classical correlation which appears when a pure quantum state is observed only partially. The two position variables x and y are classically correlated (mixed state) since one is ignoring the spin orientation (x, y, σ_z form an EPR-triplet). This experiment is practically irreversible since the second stage of the measurement (classicization) occurs when *detecting* the particle after the second field gradient. Whenever no detector is placed after the field gradient, the "measurement" is easily reversible, but in that case it has not been observed by a macroscopic observer.

2. Quantum eraser

The quantum eraser experiment (see [25]) provides a nice demonstration of how the first stage (von Neumann measurement, or "tagging") can be reversed. Several versions of this experiment have been performed. However, we restrict ourselves here to an idealized such experiment for convenience.

An eraser experiment can be visualized as a two-slit experiment using a beam of horizontally polarized photons (see Fig. 8). This beam is subsequently split in a crystal. When the split beams recombine, they produce the well-known interference pattern. However, a polarization rotator placed on, say, the left path (so that the polarization of one of the split beams–the left one–is changed from horizontal to vertical) will cause the interference pattern to vanish. This is in agreement with Feynman's rule: the paths are distinguishable since a photon traveling via the left path is vertically polarized at the screen, while a photon traveling along the right one remains horizontal. The standard explanation is that providing the "which-path" information precludes the existence of interference. The quantum eraser idea is that this which-path information can be erased, by inserting a polar-

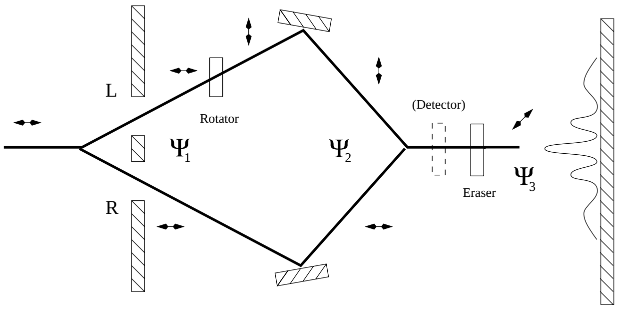


FIG. 8. Setup for the "quantum eraser" in the two-slit experiment. The detector in front of the eraser is not part of the standard setup, and illustrates the impossibility of storing the information before erasure.

ization filter aligned on the *diagonal* direction between the *recombined* beams and the screen. Such a procedure makes it impossible to tell whether a photon was horizontally or vertically polarized beforehand. Accordingly, the interference pattern on the screen is resurrected.

We start with a pure beam of horizontally polarized photons (see Fig. 8). After the splitting of the beam, the quantum state of the photon is described by the state vector

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}\Big(|L\rangle + |R\rangle\Big)|H\rangle\ , \tag{A5}$$

a function of two dichotomic variables: a location variable $\lambda = L$ (left) or R (right), and a polarization variable $\sigma = H$ (horizontal) or V (vertical). This describes a superposition of a left-photon and a right-photon after the splitting of the beams. The polarization rotator placed on the left path represents the first stage of the measurement: it can be viewed as a "tagging" operation (the left path is tagged with a vertically polarized photon and conversely) resulting in the state

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|L,V\rangle + |R,H\rangle\Big)\ . \tag{A6}$$

The crucial point is that, after tagging, the location and polarization variables are *entangled* and form an EPR-pair. Assuming, as is usually done, that the photon is either on the left path (with a vertical polarization) or on the right path (with a horizontal polarization) is classical intuition but decidedly *wrong*. We cannot witness *classical* correlation between location (L or R) and polarization (H or V); rather, the variables are *entangled* (or super-correlated) carrying negative conditional entropies ensuring that the total entropy vanishes. Indeed, the state $|\Psi_2\rangle$ is still a pure state, since it evolved from $|\Psi_1\rangle$ by a unitary transformation. At this stage, measuring the location λ of the photon (ignoring its polarization σ) yields a random variable (ignoring half of the EPR-pair gives a mixed state with positive entropy). Equivalently, measuring the polarization σ of the photon after recombining the beams (ignoring the phase hidden in the location variable λ) also yields a random variable. However, in both cases, this positive entropy is exactly compensated by a negative conditional entropy such as to preserve an overall vanishing entropy. Location and polarization play the role of conjugate (or incompatible) variables that cannot be measured simultaneously. The entanglement in $|\Psi_2\rangle$ is responsible for the loss of coherence in the location variable (the marginal density matrix of λ is a mixture) which results in the disappearance of the interference pattern. This is obvious since the cross-terms in the square of $|\Psi_2\rangle$ vanish because $|V\rangle$ and $|H\rangle$ are orthogonal.

Yet, it can be seen easily that the eraser (the diagonally oriented polarization filter placed in front of the screen) reverses the "tagging" operation, so that the quantum state $|\Psi_2\rangle$ evolves back to a pure state

$$|\Psi_3\rangle = \frac{1}{2\sqrt{2}}\Big(|L\rangle + |R\rangle\Big)\Big(|H\rangle + |V\rangle\Big) \tag{A7}$$

proportional to $|\Psi_1\rangle$, up to a trivial rotation of the polarization vector. This resuscitates the interference pattern as the location variable is now *unentangled*. Indeed, the square of the wavefunction at position x on the screen is

$$|\Psi_3|^2 = \frac{1}{4}\Big(|\psi_L(x)|^2 + |\psi_R(x)|^2 + 2\mathrm{Re}\left[\psi_L^\star(x)\psi_R(x)\right]\Big) \tag{A8}$$

where $\psi_L(x) = \langle x|L\rangle$ for example. The quantum eraser experiment only concerns the first stage of the measurement, that is the *possibility* of observing a measurement. As explained earlier, only the latter can be reversed in practice, whereas the macroscopic recording of the polarization is (practically) irreversible. An attempt at recording the polarization σ of the photon after recombination but *before* erasure (see Fig. 8) to cheat the eraser into delivering an interference pattern *and* which-path information, involves *entangling* the polarization with an ancilla A with eigenstates $|h\rangle$ and $|v\rangle$:

$$|\Psi_2'\rangle = \frac{1}{\sqrt{2}}\Big(|L,V,v\rangle + |R,H,h\rangle\Big)\ . \tag{A9}$$

Such an action is enough to thwart any attempt at recovering the interference pattern. Indeed, the action of the eraser on $|\Psi_2'\rangle$ yields

$$|\Psi_3'\rangle = \frac{1}{2\sqrt{2}}\Big(|L,v\rangle + |R,h\rangle\Big)\Big(|H\rangle + |V\rangle\Big)\ , \tag{A10}$$

leaving the location variable λ *entangled* with A (which is typically a macroscopic number of internal variables which are classically correlated when ignoring λ). In contrast with $|\Psi_3\rangle$, $|\Psi_3'\rangle$ does not give rise to an interference pattern, as it is completely analogous to Eq. (A6).

The present discussion illustrates Feynman's rule stating that, in the case of a double-slit experiment, a quantum state behaves as a particle whenever which-path information is extracted, and as a wave otherwise. As we saw above, which-path information is obtained by entangling the location variable λ. This operation by itself generates the appearance of a mixed state (and commensurate particle-like behavior) from a pure state (with wave-like behavior).

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