CAVES

More on entropy

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Expand a vector on a D-dimensional vector space as

$$|\psi\rangle = \cos\xi|\mathbf{n}\rangle + \sin\xi|\eta\rangle \;. \tag{1}$$

Here

$$|\mathbf{n}\rangle = e^{-i\varphi/2}\cos(\theta/2)|\psi_1\rangle + e^{i\varphi/2}\sin(\theta/2)|\psi_2\rangle \tag{2}$$

is the Bloch state specified by unit vector $\mathbf{n}$ (i.e., by spherical polar coördinates θ and φ) in the two-dimensional Hilbert space spanned by orthonormal vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, and $|\eta\rangle$ is a normalized vector in the orthogonal complement of the two-dimensional Bloch space. The coördinate ξ is like a polar angle between $|\psi\rangle$ and the two-dimensional Bloch space. It is not impossible to work out the Hilbert-space volume element (although many mistakes can be made in doing so) in terms of these coördinates,

$$d\mathcal{V}_D = \cos^3\xi\,(\sin\xi)^{2D-5}d\xi\,\frac{1}{4}d\mathcal{S}_2\,d\mathcal{S}_{2D-5}\;, \tag{3}$$

where $d\mathcal{S}_2 = \sin\theta\,d\theta\,d\varphi$ is the standard area element on the two-dimensional Bloch sphere (the factor of $\frac{1}{4}$ takes into account the fact that Hilbert-space angles are half the angle on the Bloch sphere, so that effectively the Bloch sphere has radius $\frac{1}{2}$) and $d\mathcal{S}_{2D-5}$ is the standard "area" element on a $(2D-5)$-dimensional sphere.

Let $V_D(\Theta)$ be the region occupied by all vectors that have $\theta \le \Theta$. The volume element in hand, one can integrate to find the volume of $V_D(\Theta)$ to be

$$\mathcal{V}_D(\Theta) = \mathcal{V}_D\sin^2(\Theta/2)\;, \tag{4}$$

where $\mathcal{V}_D = \pi^{D-1}/(D-1)!$ is the volume of Hilbert space. Consider the density operator

$$\hat\rho = \int_{V_D(\Theta)}\frac{d\mathcal{V}_D}{\mathcal{V}_D(\Theta)}|\psi\rangle\langle\psi| \tag{5}$$

that comes from averaging uniformly over $V_D(\Theta)$. It is easy to convince oneself that

$$\hat\rho = \lambda_1|\psi_1\rangle\langle\psi_1| + \lambda_2|\psi_2\rangle\langle\psi_2| + \frac{1-\lambda_1-\lambda_2}{D-2}\big(\hat 1 - |\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|\big)\;, \tag{6}$$

where the eigenvalues λ_1 and λ_2 are given by

$$\lambda_1 = \frac{2}{D}\mu_0\;,\qquad \lambda_2 = \frac{2}{D}(1-\mu_0)\;, \tag{7}$$

$$\mu_0 = \frac{1}{2}\big(1+\cos^2(\Theta/2)\big) = 1 - \frac{1}{2}\sin^2(\Theta/2)\;. \tag{8}$$

Notice that $\lambda_1 + \lambda_2 = 2/D$, which means that the eigenvalues in the orthogonal complement, all of which are equal to

$$\frac{1 - \lambda_1 - \lambda_2}{D - 2} = \frac{1}{D} , \tag{9}$$

are independent of Θ. The resulting entropy,

$$\begin{aligned} H'_D(\Theta) &= -\mathrm{tr}(\hat\rho \log \hat\rho) \\ &= -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 - (1 - \lambda_1 - \lambda_2) \log\left(\frac{1 - \lambda_1 - \lambda_2}{D - 1}\right) \\ &= \log D - \frac{2}{D}(1 + \mu_0 \log \mu_0 + (1 - \mu_0)\log(1 - \mu_0)) , \end{aligned} \tag{10}$$

is always within $2/D$ of $\log D$.

Our primary objective is to compare the entropy of the density operator (5) when $V_D(\Theta)$ occupies half the volume of Hilbert space, i.e., when $\Theta = \pi/2$ ($\mu_0 = \frac{3}{4}$), with the entropy of a sphere that occupies half the volume of Hilbert space. The entropy of the density operator (5) becomes in this case

$$H'_D = \log D - \frac{2(-1 + \frac{3}{4}\log 3)}{D} = \log D - \frac{0.3774}{D} . \tag{11}$$

For comparison, it is useful to use the large-D approximation to the entropy of a sphere of radius Φ,

$$H_D(\Phi) = \log D - \frac{1}{D}((1 + \Delta I \ln 2)\log(1 + \Delta I \ln 2) - \Delta I) , \tag{12}$$

where $\Delta I = -(D-1)\log(\sin^2\Phi)$ is the information to specify a sphere of angle Φ. The case of interest, i.e., the sphere occupying half the volume of Hilbert space, corresponds to $\Delta I = 1$ and thus to an entropy

$$H_D = \log D - \frac{(1 + \ln 2)\log(1 + \ln 2) - 1}{D} = \log D - \frac{0.2863}{D} . \tag{13}$$

The difference,

$$H_D - H'_D = \frac{0.0911}{D} , \tag{14}$$

shows that, indeed, the sphere has the larger entropy, by about a tenth of $1/D$.